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SPLITTING ALGORITHM IN KOLMOGOROV-FISHER TYPE REACTION-DIFFUSION TASK

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ABSTRACT

Analyses show that the central place in the mathematical description of the spatial-temporal dynamics of one or several interacting populations take the task of research of interaction of the migration population processes with demographic. Population models that take into account only the demographic processes are well known and fairly well-developed [3,4]. This so-called *point model*, the main assumption which is the assumption of "infinitely fast" stirring of individuals in a given area. This assumption is true if modeled area is quite small compared with the mean free path of individuals, or, equivalently, with a radius of individual activity [3]. If this provision violated, in population models must account migration.

KEYWORDS: Biological Population, Nonlinear Model, Parabolic Type Equation, Self, Similar Solutions, Lower and Upper Solutions

INTRODUCTION

The simplest and most widely used in the present situation is the hypothesis of randomness "walk" of individuals in space. This assumption allows substantiating of using as a tool of modeling equations of *reaction-diffusion* type, where as a reaction part was used the right part of the point models, and *diffusion coefficients* (mobility of individuals) are assumed to be constant.

In the framework of these models it is possible to explain such effects as wave propagation of population at settling the area and existence of complex spatio-temporal dynamics of population.

Consider in $Q_T = [0,T] \times R^N$ generalized reaction – diffusion task of Kolmogorov – Fisher type in the following form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(D\left|\frac{\partial u^m}{\partial x}\right|^{p-2} \frac{\partial u^m}{\partial x}\right) + k(t)u(1-u^{\beta}), \tag{1}$$

$$u|_{t=0} = u_0(x), x \in R^N, \sup_x u_0(x) < +\infty, \tag{2}$$

which describes the process of biological populations with double nonlinearity, which diffusion coefficient is equal to $Du^{m-1}\left|\frac{\partial u^m}{\partial x}\right|^{p-2}$. Here D, $m, p, \beta \geq 1$ - are given constants,

$$mes \sup pu_0(x) < +\infty, \left|\frac{\partial u^m}{\partial x}\right|^{p-2} \frac{\partial u^m}{\partial x} \in C(R^N)$$

Equation (1) is a generalization of the simplest diffusion model for the logistic model of population growth [1]. It can be regarded as the equation of nonlinear filtration, thermal conductivity in the presence of simultaneous influence of a source and acquisitions.

This equation in the case of m=0, k(t)= k >0 constant qas suggested (as well as the cubic equation instead quadratic nonlinearity in the right side) Fisher [3] as a stochastic model of the deterministic version of the gene in a favorable distribution of diploid populations. He examined in detail the equation and obtained some useful results. Heuristic and based on genetics conclusion of the equation have also led by A. N. Kolmogorov, I. G. Petrovskii and N.S. Piskunov, classical work [4] which served as the basis for a more rigorous analytical approach to the Fisher equation.

Properties of the solution of the Cauchy problem and the solution of equation (1) in the case of a homogeneous environment, when (1) $m = 0$ and *k(t)=k* - constant studied in detail by many other authors (see for example [3], where you can find links to other work). Suppose that in (1) $m = 0$, $k(t) = k$ -constant. In this case, for speed of propagation of the wave with wave solutions of (1)

$$Df'' - cf' + f\left(1 - f^{\beta}\right) = 0$$

$$u(t, x) = f(\xi), \ \xi = ct + x,$$

authors of [4] were rated an estimate $c \geq 2\sqrt{kD}$.

With regard to the properties of solutions of the initial task for the equation (1) is unknown and so it is interesting to trace the evolution of the process of reaction diffusion in heterogeneous environment and explore the impact of heterogeneity and case of dependences of the reaction coefficient from the time, i.e. *k*= *k*(*t*).

This paper is devoted to the study of properties of solutions of the task (1), (2). Clarified conditions on parameter m and diffusion coefficient $k(t)$, where take the place finite speed of movement of a wave. Proved also, bilateral estimation of a solution of the task (1).

We construct self-similar equations for (1)-(2)-more simple for research equations. We construct self-similar equation by the method of nonlinear splitting [1]

Below we propose an algorithm based on the splitting of the original equation of parabolic type with which assesses estimations of lower and upper solutions of the task (1)-(2).

ALGORITHM OF SPLITTING OF THE PARABOLIC TYPE EQUATION

We solve the equation

$$\bar{u}_t = k\bar{u}(1 - \bar{u}^{\beta}), \tag{3}$$

and then search solutions of the equation (1) in the form

$$u(t,x)=\bar{u}(t)w(\tau,x) \tag{4}$$

Then supplying (4) into (1) we have again equation of the form (1)

$$\frac{\partial w}{\partial \tau}=\frac{\partial}{\partial x}\left(D\left|\frac{\partial w^m}{\partial x}\right|^{p-2}\frac{\partial w^m}{\partial x}\right)+k_1(t)w(1-w^{\beta}). \tag{5}$$

where $k_1(t)=k(t)\bar{u}^{m(p-2)+m}=k(t)\bar{u}^{m(p-1)}$.

only difference is that instead of the t stands $\tau(t)=\int\left[\frac{\beta e^{m(p-2)kt}}{(1+e^{m(p-2)kt})}\right]^{\frac{m(p-2)}{\beta}}dt$, and instead of $k(t)$- $k_1(t)$.

Hence, if $mp-m=0$ equation (5) has exactly the form (1), so again, we get an equation of the form (1)

$$\frac{\partial w}{\partial t}=\frac{\partial}{\partial t}\left(D\left|\frac{\partial w^m}{\partial x}\right|^{p-2}\frac{\partial w^m}{\partial x}\right)+k(t)w(1-w^{\beta}), \tag{6}$$

however, instead of a variable by the time t stand the function $\tau(t)=\int[\bar{u}(t)]^{\frac{m(p-2)}{\beta}}dt$.

Now consider equation

$$\frac{\partial \bar{w}}{\partial \tau}=\nabla(|\nabla \bar{w}^m|^{p-2}\nabla \bar{w}^m) \tag{7}$$

Equation (7), which is called the etalon for (1), has six types of self-similar solutions, one of which has the form:

$$\bar{w}(\tau,x)=\bar{f}(\xi),\quad \xi=|x|/\tau^{1/p} \tag{8}$$

where $\bar{f}(\xi)$ satisfies the equation

$$\xi^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\left|\frac{d}{d\xi}\bar{f}^m\right|^{p-2}\frac{d}{d\xi}\bar{f}^m\right)+\frac{\xi}{p}\frac{d\bar{f}}{d\xi}=0 \tag{9}$$

at $\xi<(a/b)^{(p-1)/p}$. From (6) it is clear that if selected as the comparable function $z(t,x)=\bar{u}(t)\bar{w}(\tau,x)$ and $\bar{w}\le 1$ in Q_T, where $\bar{w}$ is solution of (6), $\bar{u}(t)$ given by (3), it will be a lower solution of the original task, if $u_0(x)\ge z(0,x)$. This result can be improved by applying again algorithm of nonlinear splitting applied above.

Now after putting $w(\tau,x)=f(\xi),\quad \xi=|x|/\tau^{1/p}$, equation (6) will turn into a self-similar equation

$$\varepsilon^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\frac{d}{d\xi}\left|f^m\right|^{p-2}\frac{df^m}{d\xi}\right)+\frac{\xi}{p}\frac{df}{d\xi}+k_1(t)(f-f^{\beta})=0 \tag{10}$$

Now if we construct a function $z(t,x)=\bar{u}(t)\bar{w}(\tau,x)$, where $\bar{w}(\tau,x)$ - upper solution of the etalon equation (7), for which runs conditions $Az\le 0$ in Q_T and $u_0(x)\le z(0,x)$, then function $z(t,x)$ will be an upper solutions of the task (1), (2). Here

$$Au=\frac{\partial u}{\partial t}-\frac{\partial}{\partial t}\left(D\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right)+k(t)u(1-u^{\beta}).$$

On these considerations are based algorithm for the construction of upper and lower solutions of the Cauchy task (1), (2), the global solvability of the initial task and definition of the critical value of the parameter β, where solution of the task becomes unlimited, and changes view of the asymptotic representation.

Let's introduce notations $Q_\infty=Q,\quad D=\{(\mathrm{t,x}):\ \mathrm{t}>0, |x|<l(t)\}$, where $l(t)>0$ at $t\ge 0$ - continuous function.

Takes place the following lemma of comparison of solutions [3,5].

Lemma. Let $u(t,x)\ge 0$ - generalized solution of the task (1), (2) and functions $u_\pm(t,x)$ such that $u_\pm\ge 0$ in Q, $u_\pm=0$ in Q\D, $u_\pm\in C^{1,2}_{t,x}(D)\cap C(\bar{D})$, $\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\in C(Q)$ and in D runs an inequality $Au_+\le 0,\quad Au_-\ge 0,\quad u_0(x)\le u_+(0,x),\quad u_0(x)\ge u_-(0,x)$ in R^N .Then $u_0(x)\le u_+(t,x),\quad u_0(x)\ge u_-(t,x),$ in Q.

Functions u_+, u_- are called, respectively, the upper and lower solution of (1), (2).

By direct calculation we can ensure that the function

$$f_0(\xi)=\left[\left(C_1-b\,|\,\xi\,|^{\frac{p}{p-1}}\right)^+\right]^{\frac{p-1}{m(p-1)-1}}, \tag{11}$$

where $C_1<0,\quad b=\frac{1}{m}\frac{m(p-1)-1}{p}a^{\frac{1}{p-1}}$, is generalized solution of the equation

$$\varepsilon^{1-N}\frac{d}{d\xi}\left(\xi^{N-1}\frac{d}{d\xi}\left|f^m\right|^{p-2}\frac{df^b}{d\xi}\right)+a\xi\frac{df}{d\xi}+aNf=0, \tag{12}$$

where constant $a>0$.

ESTIMATION OF SOLUTION OF THE TASK

Theorem 1. Let in (1)-(2) $u_0(x) \le u_+(0,x),\ x \in R^N, \xi = x/\tau^{1/p}$ and $\frac{1}{\beta - m(p-1)} - \frac{N}{p} < 0$.

Then in Q there is a global solution of (1), (2), for which we have the estimate $u(t,x) \le u_+(t,x)$, where

$$u_+(t,x) = \bar{u}(t) f_0(\xi), \tag{13}$$

here $\bar{u}(t),\ f_0(\xi)$ given by formulas (3) and (11).

Suggested that, $\beta \ge 1$, $0 \le u_0(x) \le 1$, $0 < k(t) \in C(0,\infty)$. For solution of this task is fair

Theorem 2. Let $0 \le u_0(x) \le 1 \quad x \in R^N$. Then for the solution $u(t,\ x)$ of the task (1), (2) in Q takes place an estimate

$$\bar{u}(t)(T_0+t)^{-N} e^{-\left(\frac{x}{4Dt}\right)^p} \le u(t,x) \le e^{\int_0^t k(t)dt} (T_0+t)^{-N} e^{-\left(\frac{x}{4Dt}\right)^p} \tag{14}$$

where $\bar{u}(t)$-solution of the equation (1) without diffusion part:

$$\bar{u}_t = k(t)\bar{u}(1-\bar{u}^\beta) \text{,i.e. } \bar{u}(t) = \left[\frac{\beta e^{m(p-2)kt}}{(1+e^{m(p-2)kt})}\right]^{\frac{1}{\beta}} = \left[\frac{1}{(1+e^{-m(p-2)kt})}\right]^{\frac{1}{\beta}} \le 1 \tag{15}$$

and constant $T_0 \ge 1$.

Study various properties of the solution of the Kolmogorov-Petrovsky-Piskunov-Fisher task with diffusion with reaction applied to the study of biological populations attracts the attention of many researchers, as it will be of interest from the point of view of mathematics as a nonlinear, which has an important application task. Therefore it's important generalizate this task on different cases.

In particular, it has not been studied in case of a heterogeneous environment (diffusion coefficient is a function of the spatial variable), reaction coefficient depends on time or has a more complicated character. Consider the task of reaction diffusion of Kolmogorov – Fisher type

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(|x|^n \left|\frac{\partial u^m}{\partial x}\right|^{p-2} \frac{\partial u^m}{\partial x}\right) + k(t)u(1-u^\beta) \tag{16}$$

$$u|_{t=0} = u_0(x), x \in R^N, \sup_x u_0(x) < +\infty, \tag{17}$$

describing the process of diffusion-reaction in heterogeneous (anisotropic) environment with heterogeneity coefficient x^n, where $0 \ne n \in R$.

Propose an algorithm for constructing upper solutions of the Cauchy problem (16),(17) by the method of nonlinear splitting and etalon equations.

For the solutions of the Cauchy task (16),(17) fair following theorem:

Theorem 3: Suppose that in (16) performed following conditions

$u(0,x) \le z(0,x)$, where $z(t,x) = \bar{u}(t) \cdot \bar{f}(\xi)$,

$\bar{u}(t)$ is a solution of the equation $\bar{u}_t = k(t)\bar{u}(1-\bar{u}^{\beta})$,

$$\bar{f}(\xi) = \left(a - b\xi^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{m(p-1)+n-1}}$$, where $\xi = \dfrac{|x|}{[\tau(t)]^{\frac{1}{p}}}$, $a>0$, $b = \dfrac{m(p-1)+n-1}{p^{\frac{p}{p-1}}}$,

here $\tau(t) = \int [\bar{u}(t)]^{m(p-1)+n-1} dt$.

Then the problem (16), (17) is globally solvable and for the solution in the field of Q take place following estimation $u(t,x) \le z(t,x)$.

Presence of coefficient $|x|^n$ in (16) makes difficult of study the task (16), (17) in terms of building self-similar or some partial solutions, as well as numerical modeling. Turning to the new variables, assuming $u(t,x) = w(t,\varphi(x))$ and choosing $\varphi(x)$, reduce (16) to a form which in the case $\beta \ge 1$ previously more detail studied by many authors [1], [2], [5].

NUMERICAL SCHEMES AND METHODS OF SOLVING

To solve numerically the task (1) - (2) in the $\bar{\Omega}$ let's build uniform grid $\bar{\omega}_h$ by x with step h :

$$\bar{\omega}_h = \{x_i = ih, \quad h>0, \quad i = 0,1,\dots,n, \quad hn = b\},$$

and temporary grid

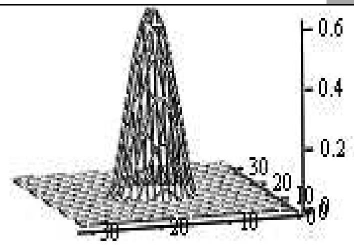
$$\bar{\omega}_\tau = \{t_j = j\tau, \quad \tau>0, \quad j = 0,1,\dots,m, \quad \tau m = T\},\ T>0.$$

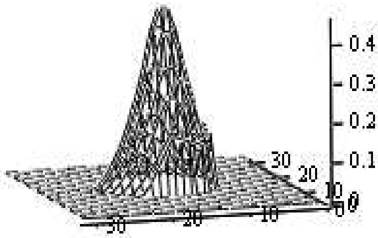
Replace the task (18)-(20) by using balance method with implicit difference scheme and receive differential task with error $O(h^2+\tau)$.

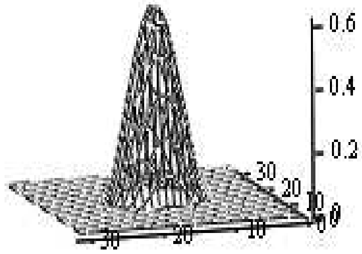
Now we present the results of numerical experiments. Iterative process is based on the method of Picard and Newton.

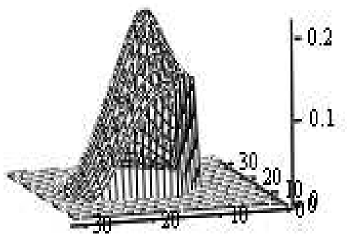
Results of computational experiments show that all iterative methods are suitable for the constructed schemes. Below listed results of numerical experiments at $k(t,x) = k(t)$ in a heterogeneous environment in two-dimensional case, for different values of parameters included in the equation (Fig.1):

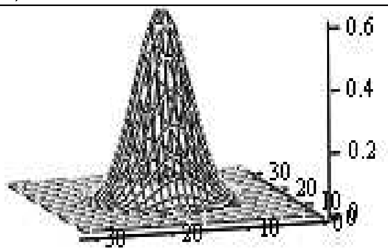
$$t \in [0,1];\ x_1 \in [-6,6];\ x_2 \in [-6,6];\ n_{x_1} = 30;\ n_{x_2} = 30;\ n_t = 100$$

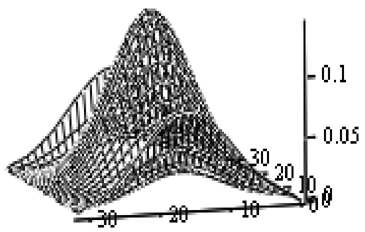
<table>
<tr><th>Values of Parameters</th><th>Results of Computational Experiment at the Initial Moment of the Time</th><th>Results of Computational Experiment at the Final Moment of the Time</th></tr>
<tr><td>$m(p-1)+n-1>0$
$m=0.7,\quad \beta=1.5$
n= 3.1, p=4.1
$eps=10^{-3}$
Average It1=1,43
Average It2=1
Average It=2,43</td><td>

time(1)</td><td>

time(100)</td></tr>
<tr><td>$m(p-1)+n-1=0$
$m=0.5,\quad \beta=1.5$
n=1, p=0.5
$eps=10^{-3}$
Average It1=2,19
Average It2=1,51
Average It=3,70</td><td>

time(1)</td><td>

time(100)</td></tr>
<tr><td>$m(p-1)+n-1<0$
$m=0.1,\quad \beta=1.5$
n=1, p=0.1
$eps=10^{-3}$
Average It1=2,02
Average It2=1,21
Average It=3,23</td><td>

time(1)</td><td>

time(100)</td></tr>
</table>

Figure 1: Results of Numerical Experiments in the Two-Dimensional Case

REFERENCES

1. Aripov M. comparison equation method for solving nonlinear boundary value problems. Tashkent. Fan. 1988. p. 137.
2. Aripov MM Sadullayev Sh.A. On the behavior of the free boundary for the equations of nonlinear filtering. / / J. Proceedings of the universities, a series of Mathematics and nat. 2003. № 1-2. pp.88- 92.
3. Marie J. Nonlinear diffusion equations in biology. Wiley, New York, 1983, 397 p.

4. Belotelov NV, Lobanov A.I.Populyatsionnye models with nonlinear diffusion. / / Mathematical modeling. -M ., 1997, № 12 , pp. 43-56 .
5. Samara AA, SP Kurdyumov, AP Mikhailov, VA Galaktionov regimes for quasilinear parabolic equations. Nauka, Moscow, 1987, 487 p.